Analysis 1A — Supplementary Paper 2020

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# Introduction

Here are the solutions to the past paper discussed in the revision session on 9th January 2023. This is designed as a guide to how much to write in the exam, and how you might want to style your solutions. To return to the homepage, click [here](http://caj50.github.io/tutoring.html).

# Question 1

Question:

For each of the following concepts, give an example that satisfies the definition and an example that does not. (You need not give any proofs.)

1. A Cauchy sequence.
2. A decreasing sequence.
3. A sequentially continuous function.
4. A conditionally convergent series.
5. An interval.

Solution.

1. An example is the sequence , where . A non-example is the sequence where .
2. An example is the sequence , where . A non-example is the sequence where .
3. An example is the function defined by . A non-example is the function , where
4. An example is the series A non-example is .
5. An example of an interval is the set . A non-example is the set

(If question 1 is like this in the exam, examples which can be used in more than one part will help you save time!)

# Question 2

Question:

The following statements paraphrase theorems, corollaries, propositions, or lemmas from the lectures. Identify them by their names.

1. Let be a real sequence. If , then there exists a sequence such that
2. and there exists such that
3. Suppose that and are two real sequences such that
4. for all . If for all and as , then converges.
5. Let and be two sequences such that
6. Then, there exists such that
7. Suppose that and . Then

Solution.

1. This is the *Bolzano-Weierstrass* theorem.
2. This is the *Leibniz alternating series test* for series.
3. This is the *Archimedian Postulate*.
4. This is the *Nested Intervals Theorem*.
5. This is the *Binomial inequality*.

# Question 3

Question:

Let be a real sequence and .

1. Show that if and only if as .
2. Assuming that , show that does *not* diverge to
   1. Use the growth factor test to show that
   2. You may use without proof that exists.
   3. Show that there exists such that
   4. for all .

In the following questions (d) and (e), you may use any result from the lectures without proof.

1. Find
2. Show that

Solution.

1. We have that Setting in this last statement gives as required.
2. We claim that To this end, fix . Since , we know that there exists such that
3. Now, for all Hence, since was arbitrary, we conclude that In particular, does not diverge to
   1. Setting for , we see that , and Since , and as , we find by the algebra of limits that
   2. Since , we find by the growth factor test that
   3. as required.
   4. By part i) and the definition of convergence, we know that such that ,
   5. Also, note that since for all
   6. Hence, for all
   7. as required.
4. First, note that via completing the square, Now, we claim that
5. To show this, we fix and consider for
6. We then have that
7. By the Archimedian Postulate, we know there exists such that Hence, for any
8. Therefore, as was arbitrary,
9. Returning to (\*) and applying the algebra of limits, we find that as ,
10. Since , we write where . This gives Rearranging, we find
11. Now, since and as , by the sandwich theorem. Hence, by the algebra of limits,
12. as required.

# Question 4

Question:

In this question, you may use any result from the lectures without proof.

1. Let . Using the theorem on the Cauchy product of series, or otherwise, show that
2. Find the radii of convergence of the following power series.
3. Say whether or not the following series converge and explain your reasoning.

Solution.

1. Recall that for , the sum is a geometric series with
2. As , this series is absolutely convergent, so the Cauchy multiplication theorem gives that
3. where for
4. Hence
5. as required.
   1. Writing
   2. with , we calculate the radius of convergence, , as
   3. Writing
   4. with , we calculate
   5. Hence, by Cauchy-Hadamard, the radius of convergence for this power series is .
   6. Writing
   7. with , we calculate
   8. Setting we claim that First, as for all , , we see that . This means that
   9. Moreover, taking the subsequence , where , we see that as
   10. So, , from which we conclude that Hence, by Cauchy-Hadamard, the radius of convergence for this power series is given by
   11. Recall from lectures that the series
   12. Since , and , we know that the series diverges.
   13. First, note that
   14. Now, setting , we find
   15. Hence, by the algebra of limits,
   16. as . So, by d’Alembert’s ratio test, we find that
   17. Finally, by the comparison test as applied to (\*\*), we conclude that
   18. Setting , we define for
   19. Using the result stated in part i), we know that as diverges, diverges. Hence, by the Cauchy condensation test, the given series diverges.